Intertwined Hamiltonians in Two Dimensional Curved Spaces

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Abstract

The problem of intertwined Hamiltonians in two dimensional curved spaces is investigated. Explicit results are obtained for Euclidean plane,Minkowski plane, Poincaré half plane (AdS_2), de Sitter Plane (dS_2), sphere, and torus. It is shown that the intertwining operator is related to the Killing vector fields and the isometry group of corresponding space. It is shown that the intertwined potentials are closely connected to the integral curves of the Killing vector fields. Two problems of considered as applications of the formalism presented in the paper. The first one is the problem of Hamiltonians with equispaced energy levels and the second one is the problem of Hamiltonians whose spectrum are like the spectrum of a free particle.

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1 Introduction

The *Intertwining* relationship between differential operators is widely studied in recent years [1, 2, 3]. This is simply the following relationship between three differential operators H_1, H_2 and $\mathcal{L}$

$$\mathcal{L}H_1 = H_2\mathcal{L} \ . \tag{1}$$

We call H_1 and H_2 the intertwined Hamiltonians and $\mathcal{L}$ the intertwining operator. One also calls Eq. (1) a *Darboux Transformation*. In this case we say that H_2 is Darboux transform of H_1 by $\mathcal{L}$. Intertwined Hamiltonians are essentially isospectral: if ψ is an eigenfunction of H_1 with eigenvalue $E \neq 0$, then $\mathcal{L}\psi$ is an eigenfunction of H_2 with the same eigenvalue. When H_1 and H_2 are Hermitian, Eq. (1) immediately results in

$$[H_1, \mathcal{L}^\dagger\mathcal{L}] = [H_2, \mathcal{L}\mathcal{L}^\dagger] = 0 \ . \tag{2}$$

Therefore $\mathcal{L}^\dagger\mathcal{L}$ and $\mathcal{L}\mathcal{L}^\dagger$ are dynamical symmetries of H_1 and H_2 respectively. It is worth noting that the above mentioned properties are in general regardless of dimension and form of Hamiltonians.

Eq. (1) appears in various fields in Physics and Mathematics. In supersymmetric Quantum Mechanics the intertwining relation is used to find exact solutions of Schrödinger equation for shape invariant potentials and to construct new exactly solvable potentials from the known ones [4]. It is also appears in the algebraic relations of topological symmetries which are some generalizations of supersymmetry [10, 11, 12].

In AdS/CFT correspondence, intertwining relations are used to realize the equivalence between the representations describing the bulk fields and the boundary fields [7].

As Eq. (1) preserves many spectral properties of the Hamiltonian, it has been widely used in the investigation of soliton equations [8] and in the study of bispectral property [9].

In this paper we investigate the general solutions of Eq. (1) in a 2D curved space. Classical and Quantum dynamics on a 2D curved space is widely studied [13, 14]. In this paper we study properties of intertwined Hamiltonians in such spaces. The complete exact solutions are obtained for Euclidean plane, Minkowski plane, Poincarè half plane, de Sitter plane, Sphere, and Torus. It is shown that the solutions are related to Killing vector fields and isometry group of corresponding space. Our solutions are also closely related to the problem of superintegrability in 2D spaces [1, 2, 15, 16].

Consider a two dimensional curved surface with coordinates x^a, $a = 1, 2$ and metric g_{ab}. Suppose that H_1 and H_2 are two Hamiltonians defined as below

$$H_1 = -\nabla^2 + V_1(x^1, x^2) \ , \tag{3}$$

$$H_2 = -\nabla^2 + V_2(x^1, x^2) \ , \tag{4}$$

where ∇^2 is the Laplasian operator, $\nabla^2 = \frac{1}{\sqrt{g}}\partial_a\left(\sqrt{g}g^{ab}\partial_b\right)$, and $g = \det(g_{ab})$ and g^{ab} is the inverse of g_{ab}, i.e. $g^{ac}g_{cb} = \delta^a_b$. We call V_1 and V_2 intertwined potentials if Eq. (1) holds. In this paper we will take $\mathcal{L}$ to be a first order differential operator,

$$\mathcal{L} = L_0 + L^a\partial_a \equiv L_0 + \mathbf{L} \ , \tag{5}$$

where L_0 and L^a, $a = 1, 2$, are real valued functions of x^1 and x^2. Our aim is to find L_0, L^a, V_1, and V_2 such that Eq. (1) holds. Substituting $\mathcal{L}$, H_1 and H_2 from Eqs. (5), (3), and (4) into Eq. (1), we arrive at the following equations

$$L^c \partial_c g^{ab} - g^{bc} \partial_c L^a - g^{ac} \partial_c L^b = 0 \ , a, b = 1, 2 \ , \tag{6}$$

$$PL^a = 2g^{ab} \partial_b L_0 \ , a = 1, 2 \tag{7}$$

$$PL_0 = \nabla^2 L_0 + L^c \partial_c V_1 \ , \tag{8}$$

where $P = V_2 - V_1$. Eqs. (6) – (8) are completely general, i.e. they are correct in all dimensions. In this paper we will investigate only two dimensional spaces. Eq. (6) is the well known Killing equation. Its solutions are the Killing vector fields of the corresponding space. The rest of the paper is devoted to solve Eqs. (6) – (8) in some two dimensional spaces, namely Euclidean plane, Minkowski plane, Poincarè half plane, de Sitter plane, sphere, and torus. Finally we give two examples as applications of the formalism presented in this paper. The first one is the problem of Hamiltonians with equispaced energy levels on two dimensional surfaces and the second one is about the particles which move in a nontrivial potential, but their spectrum are like free particle spectrum. We call these particles *free like* particles.

2 Euclidean Plane

Euclidean plane the simplest case in which the metric is $g_{ab} = diag(1, 1)$. We use the notation $x^1 = x$, $x^2 = y$. Then Eq. (6) takes the following form

$$\partial_x L^x = 0 \ , \tag{9}$$

$$\partial_y L^y = 0 \ , \tag{10}$$

$$\partial_x L^y + \partial_y L^x = 0 \ , \tag{11}$$

The above equations has a general solution, $\mathbf{L} = L^x \partial_x + L^y \partial_y$, of the form

$$\mathbf{L} = \alpha \mathbf{L}_1 + \beta \mathbf{L}_2 + \gamma \mathbf{L}_3 \ , \tag{12}$$

where α, β and γ are real constants and

$$\mathbf{L}_1 = \partial_x \ , \tag{13}$$

$$\mathbf{L}_2 = \partial_y \ , \tag{14}$$

$$\mathbf{L}_3 = y\partial_x - x\partial_y \ . \tag{15}$$

It is easily seen that $\mathbf{L}_1$, $\mathbf{L}_2$ and $\mathbf{L}_3$ satisfy the algebra of the isometry group of the Euclidean plane, namely $E(2)$,

$$[\mathbf{L}_1, \mathbf{L}_2] = 0, \quad [\mathbf{L}_2, \mathbf{L}_3] = \mathbf{L}_1, \quad [\mathbf{L}_3, \mathbf{L}_1] = \mathbf{L}_2 \ , \tag{16}$$

As the general solution for $\mathbf{L}$ is a combination of $\mathbf{L}_1$, $\mathbf{L}_2$ and $\mathbf{L}_3$, we classify the solutions in three classes as follows. It is worth noting that from a Physical (or geometrical) point of view these classes are not independent. For example class 1 and class 2 below are equivalent. Therefore this classification is from an algebraic point of view which means how one can construct the most general intertwining operator.

2.1 Class 1

In this class we take $\mathbf{L} = \mathbf{L}_1 \equiv \partial_x$. taking this solution for $\mathbf{L}$, one can easily see that Eq. (7) leads to

$$P = 2\partial_x L_0 \ , \tag{17}$$

$$\partial_y L_0 = 0 \ , \tag{18}$$

Eq. (18) means that L_0 is a function of x only, therefore if we choose L_0 arbitrarily, we can find P easily:

$$L_0 = L_0(x) \ , \tag{19}$$

$$P = 2L_0' \tag{20}$$

where $L_0'(x) = dL_0/dx$ (Throughout this paper a '$\prime$' sign means the derivative of a function with respect to it's argument).

Next we use Eq. (8) to find V_1. Inserting the above results in Eq. (8) we get

$$2L_0L_0' = L_0'' + \partial_x V_1 \ . \tag{21}$$

This equation is easily integrated and one arrives at the following general solution for V_1 and V_2.

$$V_1(x,y) = L_0^2(x) - L_0'(x) + f(y) \ , \tag{22}$$

$$V_2(x,y) = L_0^2(x) + L_0'(x) + f(y) \ , \tag{23}$$

where $f(y)$ is an arbitrary function of y.

2.2 Class 2

In this class we take $\mathbf{L} = \mathbf{L}_2 = \partial_y$. This class is similar to class 1. The solutions can be obtained easily from the class 1 solutions by interchanging the role of x and y

$$L_0 = L_0(y) \ , \tag{24}$$

$$P = 2L_0' \tag{25}$$

$$V_1(x,y) = L_0^2(y) - L_0'(y) + f(x) \ , \tag{26}$$

$$V_2(x,y) = L_0^2(y) + L_0'(y) + f(x) \ . \tag{27}$$

2.3 Class 3

In this class we take $\mathbf{L} = \mathbf{L}_3 \equiv y\partial_x - x\partial_y$. With this choice for $\mathbf{L}$, Eqs. (7) and (8) take the following form

$$Py = 2\partial_x L_0 \ , \tag{28}$$

$$-Px = 2\partial_y L_0 \tag{29}$$

$$PL_0 = \partial_x^2 L_0 + \partial_y^2 L_0 + y\partial_x V_1 - x\partial_y V_1 \ , \tag{30}$$

These equations are easily solved with the following change of variables

$$u = \frac{x}{y} \ , v = x^2 + y^2 \ . \tag{31}$$

The general solutions are

$$L_0 = L_0(u) \ , \tag{32}$$

$$P = \frac{2}{y^2} L_0' \tag{33}$$

$$V_1(u,v) = \frac{1}{v}\left(L_0^2(u) - (1+u^2)L_0'(u) + f(v)\right) \ , \tag{34}$$

$$V_2(u,v) = \frac{1}{v}\left(L_0^2(u) + (1+u^2)L_0'(u) + f(v)\right) \ . \tag{35}$$

The meaning of the new variables u and v is worth noting. The family of curves given by $v = const$ is the family of integral curves of the Killing vector field $\mathbf{L} = y\partial_x - x\partial_y$. Also the curves given by $u = const$ are normal to the curves given by $v = const$.

3 Minkowski plane

For Minkowski plane the metric is $g_{ab} = diag(-1,1)$. We use the notation $x^1 = t$, $x^2 = x$. Then Eq. (6) takes the following form

$$\partial_t L^t = 0 \ , \tag{36}$$

$$\partial_x L^x = 0 \ , \tag{37}$$

$$\partial_t L^x - \partial_x L^t = 0 \ , \tag{38}$$

The above equations has a general solution, $\mathbf{L} = L^t\partial_t + L^x\partial_x$, of the form

$$\mathbf{L} = \alpha\mathbf{L}_1 + \beta\mathbf{L}_2 + \gamma\mathbf{L}_3 \ , \tag{39}$$

where α, β and γ are real constants and

$$\mathbf{L}_1 = \partial_t \ , \tag{40}$$

$$\mathbf{L}_2 = \partial_x \ , \tag{41}$$

$$\mathbf{L}_3 = x\partial_t + t\partial_x \ . \tag{42}$$

It is easily seen that $\mathbf{L}_1$, $\mathbf{L}_2$ and $\mathbf{L}_3$ satisfy the algebra of the isometry group of the Minkowski plane, Which is a subgroup of Poincaré group

$$[\mathbf{L}_1, \mathbf{L}_2] = 0, \quad [\mathbf{L}_2, \mathbf{L}_3] = \mathbf{L}_1, \quad [\mathbf{L}_1, \mathbf{L}_3] = \mathbf{L}_2 \ , \tag{43}$$

As the general solution for $\mathbf{L}$ is a combination of $\mathbf{L}_1$, $\mathbf{L}_2$ and $\mathbf{L}_3$, we classify the solutions in three classes as follows

3.1 Class 1

In this class we take $\mathbf{L} = \mathbf{L}_1 \equiv \partial_t$. taking this solution for $\mathbf{L}$, one can easily see that Eq. (7) leads to

$$P = -2\partial_t L_0 \ , \tag{44}$$

$$\partial_x L_0 = 0 \ , \tag{45}$$

Eq. (45) means that L_0 is a function of t only, therefore if we choose L_0 arbitrarily, we can find P easily:

$$L_0 = L_0(t) \ , \tag{46}$$

$$P = -2L_0' \tag{47}$$

where $L_0'(t) = dL_0/dt$.

Next we use Eq. (8) to find V_1. Inserting the above results in Eq. (8) we get

$$2L_0L_0' = L_0'' + \partial_t V_1 \ . \tag{48}$$

This equation is easily integrated and one arrives at the following general solution for V_1 and V_2.

$$V_1(t,x) = -L_0^2(t) + L_0'(t) + f(x) \ , \tag{49}$$

$$V_2(t,x) = -L_0^2(t) - L_0'(t) + f(x) \ , \tag{50}$$

where $f(x)$ is an arbitrary function of x.

3.2 Class 2

In this class we take $\mathbf{L} = \mathbf{L}_2 = \partial_x$. This class is very similar to class 1. The solutions can be obtained easily in the same manner

$$L_0 = L_0(x) \ , \tag{51}$$

$$P = 2L_0' \tag{52}$$

$$V_1(t,x) = L_0^2(x) - L_0'(x) + f(t) \ , \tag{53}$$

$$V_2(t,x) = L_0^2(x) + L_0'(x) + f(t) \ . \tag{54}$$

3.3 Class 3

In this class we take $\mathbf{L} = \mathbf{L}_3 \equiv x\partial_t + t\partial_x$. With this choice for $\mathbf{L}$, Eqs. (7) and (8) take the following form

$$Px = -2\partial_t L_0 \ , \tag{55}$$

$$Pt = 2\partial_x L_0 \tag{56}$$

$$PL_0 = -\partial_t^2 L_0 + \partial_x^2 L_0 + x\partial_t V_1 + t\partial_x V_1 \ , \tag{57}$$

These equations are easily solved with the following change of variables

$$u = \frac{t}{x} \ , v = t^2 - x^2 \ . \tag{58}$$

The general solutions are

$$L_0 = L_0(u) \ , \tag{59}$$

$$P = -\frac{2}{x^2} L_0' \tag{60}$$

$$V_1(u,v) = -\frac{1}{v}\left(L_0^2(u) + (1-u^2)L_0'(u) + f(v)\right) \ , \tag{61}$$

$$V_2(u,v) = -\frac{1}{v}\left(L_0^2(u) - (1-u^2)L_0'(u) + f(v)\right) \ . \tag{62}$$

4 Poincaré half plane (AdS_2)

Poincaré half plane is a 2-dimensional Riemannian manifold with constant Gaussian curvature $\kappa = -1$. In fact it is a space with coordinates (x, y) with $y > 0$ and its metric is given by $g_{ab} = diag(1/y^2, 1/y^2)$.

In this space Eq. (6) takes the following form

$$L^y = y\partial_x L^x \ , \tag{63}$$

$$L^y = y\partial_y L^y \ , \tag{64}$$

$$\partial_x L^y + \partial_y L^x = 0 \ , \tag{65}$$

It can be easily shown that the most general solution of Eqs. (63) – (65) for $\mathbf{L} = L^x\partial_x + L^y\partial_y$ is of the form of Eq. (12) with

$$\mathbf{L}_1 = \partial_x \ , \tag{66}$$

$$\mathbf{L}_2 = x\partial_x + y\partial_y \ , \tag{67}$$

$$\mathbf{L}_3 = (x^2 - y^2)\partial_x + 2xy\partial_y \ . \tag{68}$$

These are in fact Killing vector fields of Poincare half plane. They satisfy the algebra of isometry group of this space namely $SL(2,\mathbb{R})$.

$$[\mathbf{L}_1, \mathbf{L}_2] = \mathbf{L}_1 \ , [\mathbf{L}_2, \mathbf{L}_3] = \mathbf{L}_3 \ , [\mathbf{L}_1, \mathbf{L}_3] = 2\mathbf{L}_2 \ . \tag{69}$$

Again we distinguish three classes and investigate each class separately

4.1 Class 1

In this case we take $\mathbf{L} = \mathbf{L}_1 \equiv \partial_x$. Then Eqs. (7) and (8) simplify as follows

$$P = 2y^2 \partial_x L_0 \ , \tag{70}$$

$$\partial_y L_0 = 0 \ , \tag{71}$$

$$P L_0 = y^2 (\partial_x^2 L_0 + \partial_y^2 L_0) + \partial_x V_1 \ . \tag{72}$$

From Eq. (71) one can easily see that L_0 is a function of x only and the solution for P, V_1 and V_2 are as the following

$$L_0 = L_0(x) \ , \tag{73}$$

$$P = 2y^2 L_0' \tag{74}$$

$$V_1(x, y) = y^2 \left(L_0^2(x) - L_0'(x) + f(y) \right) \ , \tag{75}$$

$$V_2(x, y) = y^2 \left(L_0^2(x) + L_0'(x) + f(y) \right) \ , \tag{76}$$

4.2 Class 2

This class is given by $\mathbf{L} = \mathbf{L}_2 \equiv x\partial_x + y\partial_y$. Then Eqs. (7) and (8) take the following form

$$P x = 2y^2 \partial_x L_0 \ , \tag{77}$$

$$P y = 2y^2 \partial_y L_0 \ , \tag{78}$$

$$P L_0 = y^2 (\partial_x^2 L_0 + \partial_y^2 L_0) + x\partial_x V_1 + y \partial_y V_1 \ , \tag{79}$$

One can easily solve these equations with the following change of variables

$$u = x^2 + y^2 \ \ , v = \frac{x}{y} \ . \tag{80}$$

The solutions are

$$L_0 = L_0(u) \ , \tag{81}$$

$$P = 4y^2 L_0' \tag{82}$$

$$V_1(u, v) = \frac{1}{1 + v^2} (L_0^2(u) - 2u L_0'(u) + f(v)) \ , \tag{83}$$

$$V_2(u, v) = \frac{1}{1 + v^2} (L_0^2(u) + 2u L_0'(u) + f(v)) \ , \tag{84}$$

Here again one can notice to the relationship between the new variables and the integral curves of the Killing vector field. In fact the family of curves given by $v = \text{const}$ is the family of integral curves of the Killing vector field $\mathbf{L}_2$. These are normal to the family of curves given by $u = \text{const}$.

4.3 Class 3

In this case we take $\mathbf{L} = \mathbf{L}_3 \equiv (x^2 - y^2)\partial_x + 2xy\partial_y$. Then Eqs. (7) and (8) read

$$P(x^2 - y^2) = 2y^2 \partial_x L_0 \ , \tag{85}$$

$$P(2xy) = 2y^2 \partial_y L_0 \ , \tag{86}$$

$$PL_0 = y^2(\partial_x^2 L_0 + \partial_y^2 L_0) + (x^2 - y^2)\partial_x V_1 + 2xy\partial_y V_1 \ , \tag{87}$$

One can easily solve these equations with the following change of variables

$$u = \frac{x^2 + y^2}{x} \quad , v = \frac{x^2 + y^2}{y} \ . \tag{88}$$

The solutions are

$$L_0 = L_0(u) \ , \tag{89}$$

$$P = \frac{2y^2}{x^2} L_0' \tag{90}$$

$$V_1(u, v) = \frac{1}{v^2}(L_0^2(u) - u^2 L_0'(u) + f(v)) \ , \tag{91}$$

$$V_2(u, v) = \frac{1}{v^2}(L_0^2(u) + u^2 L_0'(u) + f(v)) \ , \tag{92}$$

5 de Sitter Plane (dS_2)

For de Sitter plane we choose the notation $x^1 = t$ and $x^2 = x$ for coordinates. The metric is given by $g_{ab} = diag(-1/t^2, 1/t^2)$.

In this space Eq. (6) takes the following form

$$L^t = t\partial_x L^x \ , \tag{93}$$

$$L^t = t\partial_t L^t \ , \tag{94}$$

$$\partial_x L^t - \partial_t L^x = 0 \ , \tag{95}$$

It can be easily shown that the most general solution of Eqs. (93) – (95) for $\mathbf{L} = L^t\partial_t + L^x\partial_x$ is of the form of Eq. (12) with

$$\mathbf{L}_1 = \partial_x \ , \tag{96}$$

$$\mathbf{L}_2 = x\partial_x + t\partial_t \ , \tag{97}$$

$$\mathbf{L}_3 = (x^2 + t^2)\partial_x + 2xt\partial_t \ . \tag{98}$$

These are in fact Killing vector fields of de Sitter plane. They satisfy the algebra of isometry group of this space namely $SL(2, \mathbb{R})$.

$$[\mathbf{L}_1, \mathbf{L}_2] = \mathbf{L}_1 \ , [\mathbf{L}_2, \mathbf{L}_3] = \mathbf{L}_3 \ , [\mathbf{L}_1, \mathbf{L}_3] = 2\mathbf{L}_2 \ . \tag{99}$$

Again we distinguish three classes and investigate each class separately

5.1 Class 1

In this case we take $\mathbf{L} = \mathbf{L}_1 \equiv \partial_x$. Then Eqs. (7) and (8) simplify as follows

$$P = 2t^2 \partial_x L_0 \ , \tag{100}$$

$$\partial_t L_0 = 0 \ , \tag{101}$$

$$P L_0 = t^2 (\partial_x^2 L_0 - \partial_t^2 L_0) + \partial_x V_1 \ . \tag{102}$$

From Eq. (101) one can easily see that L_0 is a function of x only and the solution for P, V_1 and V_2 are as the following

$$L_0 = L_0(x) \ , \tag{103}$$

$$P = 2t^2 L_0' \tag{104}$$

$$V_1(t,x) = t^2 \left(L_0^2(x) - L_0'(x) + f(t) \right) \ , \tag{105}$$

$$V_2(t,x) = t^2 \left(L_0^2(x) + L_0'(x) + f(t) \right) \ , \tag{106}$$

5.2 Class 2

This class is given by $\mathbf{L} = \mathbf{L}_2 \equiv x\partial_x + t\partial_t$. Then Eqs. (7) and (8) take the following form

$$Px = 2t^2 \partial_x L_0 \ , \tag{107}$$

$$Pt = -2t^2 \partial_t L_0 \ , \tag{108}$$

$$P L_0 = t^2 (\partial_x^2 L_0 - \partial_t^2 L_0) + x \partial_x V_1 + t \partial_t V_1 \ , \tag{109}$$

One can easily solve these equations with the following change of variables

$$u = x^2 - t^2 \quad , v = \frac{x}{t} \ . \tag{110}$$

The solutions are

$$L_0 = L_0(u) \ , \tag{111}$$

$$P = 4t^2 L_0' \tag{112}$$

$$V_1(u,v) = \frac{1}{v^2 - 1} (L_0^2(u) - 2uL_0'(u) + f(v)) \ , \tag{113}$$

$$V_2(u,v) = \frac{1}{v^2 - 1} (L_0^2(u) + 2uL_0'(u) + f(v)) \ , \tag{114}$$

Here again one can notice to the relationship between the new variables and the integral curves of the Killing vector field. In fact the family of curves given by $v = \text{const}$ is the family of integral curves of the Killing vector field $\mathbf{L}_2$. These are normal to the family of curves given by $u = \text{const}$.

5.3 Class 3

In this case we take $\mathbf{L} = \mathbf{L}_3 \equiv (x^2+t^2)\partial_x + 2xt\partial_t$. Then Eqs. (7) and (8) read

$$P(x^2+t^2) = 2t^2\partial_x L_0 \ , \tag{115}$$

$$P(2xt) = -2t^2\partial_t L_0 \ , \tag{116}$$

$$PL_0 = t^2(\partial_x^2 L_0 - \partial_t^2 L_0) + (x^2+t^2)\partial_x V_1 + 2xt\partial_t V_1 \ , \tag{117}$$

One can easily solve these equations with the following change of variables

$$u = \frac{x^2-t^2}{x} \quad , v = \frac{x^2-t^2}{t} \ . \tag{118}$$

The solutions are

$$L_0 = L_0(u) \ , \tag{119}$$

$$P = \frac{2t^2}{x^2} L_0' \tag{120}$$

$$V_1(u,v) = \frac{1}{v^2}(L_0^2(u) - u^2 L_0'(u) + f(v)) \ , \tag{121}$$

$$V_2(u,v) = \frac{1}{v^2}(L_0^2(u) + u^2 L_0'(u) + f(v)) \ , \tag{122}$$

6 Sphere

For a sphere with unit radius we take the coordinates as $x^1 = \theta$ and $x^2 = \phi$. The metric is $g_{ab} = diag(1, \sin^2\theta)$. Then Eq. (6) implies

$$\partial_\theta L^\theta = 0 \ , \tag{123}$$

$$\tan\theta \partial_\phi L^\phi + L^\theta = 0 \ , \tag{124}$$

$$\partial_\phi L^\theta + \sin^2\theta \partial_\theta L^\phi = 0 \ . \tag{125}$$

The general solution of the above equations, $\mathbf{L} = L^\theta\partial_\theta + L^\phi\partial_\phi$, is given by Eq. (12) with

$$\mathbf{L}_1 = \partial_\phi \ , \tag{126}$$

$$\mathbf{L}_2 = \cos\phi\partial_\theta - \sin\phi\cot\theta\partial_\phi \ , \tag{127}$$

$$\mathbf{L}_3 = \sin\phi\partial_\theta + \cos\phi\cot\theta\partial_\phi \ . \tag{128}$$

Here again we obtained the Killing vector fields of S^2 (two dimensional sphere), which satisfy the algebra of isometry group of S^2, namely $SO(3)$

$$[\mathbf{L}_1, \mathbf{L}_3] = \mathbf{L}_2 \quad , [\mathbf{L}_3, \mathbf{L}_2] = \mathbf{L}_1 \quad , [\mathbf{L}_2, \mathbf{L}_1] = \mathbf{L}_3 \ , \tag{129}$$

Again we investigate each of the above solutions in a separate class. Here again one should note that there are only one class from a physical point of view. In fact all three classes given below are equivalent. But algebraically one can use these classes to construct the most general intertwining operator.

6.1 Class 1

In this class $\mathbf{L} = \mathbf{L}_1 \equiv \partial_\phi$. Then Eqs. (7) , (8) read

$$\partial_\theta L_0 = 0 \ , \tag{130}$$

$$P \sin^2\theta - 2\partial_\phi L_0 = 0 \ , \tag{131}$$

$$P L_0 = \cot\theta \partial_\theta L_0 + \partial_\theta^2 L_0 + \frac{1}{\sin^2\theta}\partial_\phi^2 L_0 + \partial_\phi V_1 \ . \tag{132}$$

The above equations result in the following general solutions

$$L_0 = L_0(\phi) \ , \tag{133}$$

$$P = \frac{2}{\sin^2\theta} L_0' \tag{134}$$

$$V_1(\theta,\phi) = \frac{1}{\sin^2\theta}(L_0^2 - L_0' + f(\theta)) \ , \tag{135}$$

$$V_2(\theta,\phi) = \frac{1}{\sin^2\theta}(L_0^2 + L_0' + f(\theta)) \ . \tag{136}$$

6.2 Class 2

In this class $\mathbf{L} = \mathbf{L}_2 \equiv \cos\phi\partial_\theta - \sin\phi\cot\theta\partial_\phi$. Then Eqs. (7) , (8) read

$$P\cos\phi = 2\partial_\theta L_0 \ , \tag{137}$$

$$P\sin\theta\cos\theta\sin\phi + 2\partial_\phi L_0 = 0 \ , \tag{138}$$

$$P L_0 = \cot\theta\partial_\theta L_0 + \partial_\theta^2 L_0 + \frac{1}{\sin^2\theta}\partial_\phi^2 L_0 + \cos\phi\partial_\theta V_1 - \cot\theta\sin\phi\partial_\phi V_1 \ . \tag{139}$$

One can easily solve these equations with the help of the following change of variables

$$u = \cos\phi\tan\theta \ \ , v = \sin\phi\sin\theta \ . \tag{140}$$

The general solutions are given below

$$L_0 = L_0(u) \ , \tag{141}$$

$$P = \frac{2}{\cos^2\theta} L_0' \tag{142}$$

$$V_1(u,v) = \frac{1}{1-v^2}(L_0^2(u) - (1+u^2)L_0'(u) + f(v)) \ , \tag{143}$$

$$V_2(u,v) = \frac{1}{1-v^2}(L_0^2(u) + (1+u^2)L_0'(u) + f(v)) \ , \tag{144}$$

where $f(v)$ is an arbitrary function of v.

6.3 Class 3

In this class $\mathbf{L} = \mathbf{L}_3 \equiv \sin\phi\partial_\theta + \cos\phi\cot\theta\partial_\phi$. Then Eqs. (7) , (8) read

$$P\sin\phi = 2\partial_\theta L_0 \ , \tag{145}$$

$$P\sin\theta\cos\theta\cos\phi - 2\partial_\phi L_0 = 0 \ , \tag{146}$$

$$P L_0 = \cot\theta\partial_\theta L_0 + \partial_\theta^2 L_0 + \frac{1}{\sin^2\theta}\partial_\phi^2 L_0 + \sin\phi\partial_\theta V_1 + \cot\theta\cos\phi\partial_\phi V_1 \ . \tag{147}$$

One can easily solve these equations with the help of following change of variables

$$u = \sin\phi \tan\theta \ \ , v = \cos\phi \sin\theta \ . \tag{148}$$

The general solution are given below

$$L_0 = L_0(u) \ , \tag{149}$$
$$P = \frac{2}{\cos^2\theta} L_0' \tag{150}$$
$$V_1(u,v) = \frac{1}{1-v^2}(L_0^2(u) - (1+u^2)L_0'(u) + f(v)) \ , \tag{151}$$
$$V_2(u,v) = \frac{1}{1-v^2}(L_0^2(u) + (1+u^2)L_0'(u) + f(v)) \ , \tag{152}$$

where $f(v)$ is an arbitrary function of v.

7 Torus

Consider a torus which it's minor and major radii are 1 and R respectively. The coordinates are $x^1 = \theta$ and $x^2 = \phi$ which show the position of minor and major radii respectively. The metric is $g_{ab} = diag\left(1, (R+\cos\theta)^2\right)$. Eq. (6) reads

$$\partial_\theta L^\theta = 0 \ , \tag{153}$$
$$(R+\cos\theta)^2 \partial_\theta L^\phi - \partial_\phi L^\theta = 0 \ , \tag{154}$$
$$(R+\cos\theta)\partial_\phi L^\phi - \sin\theta L^\theta = 0 \ , \tag{155}$$

The only solution of the above equations is $\mathbf{L} = \partial_\phi$. In fact this is the only vector field on the torus. The isometry group corresponding to this vector field is $U(1)$. Then one can easily find the solutions for L_0, P, V_1 and V_2,

$$L_0 = L_0(\phi) \ , \tag{156}$$
$$P = \frac{2}{(R+\cos\theta)^2} L_0' \tag{157}$$
$$V_1(\theta,\phi) = \frac{1}{(R+\cos\theta)^2}(L_0^2 - L_0' + f(\theta)) \ , \tag{158}$$
$$V_2(\theta,\phi) = \frac{1}{(R+\cos\theta)^2}(L_0^2 + L_0' + f(\theta)) \ , \tag{159}$$

where $f(\theta)$ is an arbitrary function of θ.

8 Applications

In this section we give two examples for the formalism given in the above sections. The first example is about quantum systems with equispaced energy levels and the second one is quantum systems which their spectrum is like the spectrum of a free particle.

8.1 Equispaced Energy Levels

Quantum systems with equispaced energy levels are of some importance in condensed matter Physics and optics [17]. In the formalism given in this paper it it is easy to investigate such systems. In fact if $P = V_2 - V_1$ be a nonzero constant, then both the Hamiltonians H_1 and H_2 have equispaced energy levels and the operator $\mathcal{L}$ will be the lowering operator for the Hamiltonian H_1, i.e. $[\mathcal{L}, H_1] = P\mathcal{L}$. This means that P is the space between energy levels. If we have a look at the solutions on surfaces investigated in this paper we will see that among these surfaces only the Euclidean plane allows for such solutions i.e. for solutions in which P is constant and consequently the energy levels are equispaced. A question arises naturally: Is the Euclidean plane the only surface that allows for the systems with equispaced energy levels? If not, then what is the general solution? To answer this question we will put $P = const.$ in Eqs. (6) – (8) and see what conditioins this choice will put on the surface. One can put $P = 1$ without loss of generality. This means that we measure energy in P units. With this choice and using Eqs. (6) and (7) one can easily verify that the Killing vector field must satisfy the following condition

$$\nabla_a L^b \equiv \partial_a L_b + \Gamma^b_{ac} L^c = 0 \; , a, b = 1, 2 \tag{160}$$

where $\nabla_a L^b$ is the covariant derivative of L^b and $\Gamma^b_{ac} := \frac{1}{2} g^{bd} \left[\partial_a g_{dc} + \partial_c g_{ad} - \partial_d g_{ac}\right]$ is the connection. Now using the definition of curvature tensor $[\nabla_a, \nabla_b]\, L^c = R_{bad}{}^c L^d$ and Eq. (160) one can easily verify that Eq. (160) is satisfied only if the curvature tensor equals to zero identically. The curvature tensor in two dimensions has just one independent component and it is $R_{1212} = \frac{1}{2} Rg$ where R is the curvature of the surface and g is the determinant of the metric tensor. This fact together with the definition of curvature tensor and Eq. (160) results in

$$\frac{1}{2} Rg(g^{12} L^1 - g^{11} L^2) = 0 \; , \tag{161}$$

$$\frac{1}{2} Rg(g^{22} L^1 - g^{21} L^2) = 0 \; . \tag{162}$$

These equations are consistent only if $R = 0$. Therefore in our formalism systems with equispaced energy levels can only exist on flat surfaces which are surfaces with zero curvature tensor. This means that for example on a compact surface like torus or sphere one can not construct a quantum system with equispaced energy levels. This is also verified from the fact that the condition $P = const.$ leads to $\nabla^2 L_0 = 0$. Because this is the Laplace equation which has a trivial solution on a compact surface.

8.2 Free Like Particles

By *free like* we mean a particle which moves in a nontrivial potential, but its spectrum is essentially the free particle spectrum. This makes sense only on a compact surface because on a compact surface the free particle spectrum is discrete. In our formalism this potential is the partner of constant potential. For a constant potential V_1 Eq. (8) leads to

$$P L_0 = \nabla^2 L_0 \; . \tag{163}$$

Using the above equation and Eq. (7) one can find L_0 and P. We construct the potentials of free like particles on sphere and torus.

Sphere: To find free like potential on a sphere one can easily put $f(\theta) = 1$ and $L_0(\phi) = \tan\phi$ in Eqs. (135) and (136). Then one gets $V_1(\theta, \phi) = 0$ and

$$V_2(\theta, \phi) = \frac{2}{\sin^2\theta\cos^2\phi} \; . \tag{164}$$

Torus: On a torus using Eqs. (158) and (159) and putting $f(\theta) = 1$ and $L_0(\phi) = \tan\phi$ one arrives at $V_1(\theta, \phi) = 0$ and

$$V_2(\theta, \phi) = \frac{2}{(\cos\theta + R)^2\cos^2\phi} \; . \tag{165}$$

9 Summary and Concluding Remarks

In this paper we investigated the intertwined Hamiltonians in some two dimensional curved spaces. We found the general forms of intertwined potentials and the intertwining operators. It was shown that the intertwining operators are closely related to the Killing vector fields of the corresponding space. In fact the intertwining operator is the Killing vector field plus a real valued function in the corresponding space. This real valued function is closely connected to the integral curves of corresponding Killing vector filed.

A comment on the case of $P \equiv V_2 - V_1 = 0$ is in order. In this case the intertwining operator $\mathcal{L}$ commutes with $H_1 = H_2$. In this case, L_0 is a constant and the potential $V_1 = V_2$ is a function of the variable 'v' only. As $v = const.$ is an integral curve of the Killing vector field, this means that in quantum systems in which the potential is constant along the integral curves of the Killing vector field, the corresponding Killing vector field is a constant of motion.

Another interesting case is the the case in which $P \equiv V_2 - V_1 = const.$ In this case the energy levels of the Hamiltonian H_1 (and H_2) are equispaced. It can be easily verified that among the spaces studied in this paper, this happens only in Euclidean plane (cases 1 and 2). We have shown that only surfaces with zero curvature allow for such solutions.

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